A Time–Frequency Application With the Stokes–Woodward Technique

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***Abstract*—In a recent paper, we have generalized Woodward's theorem and applied it to the case of random signals jointly modulated in amplitude and frequency. This generalization yields a new spectral technique to estimate the amount of energy due to mode coupling without calling for higher order statistics. Two power spectra are detected; the first is related to the independent modes, and the second contains extra energy caused by mode coupling. This detection is now extended from frequency to time–frequency domain. A comparison between a wavelet transform and our time–frequency technique shows good agreement along with new insight into the time occurrence of the nonlinearities or mode coupling. An application to water surface waves is given in this letter as an example.**

***Index Terms*—Amplitude modulation–frequency modulation (AM-FM), horizontal asymmetry, instantaneous amplitude, instantaneous frequency, mode coupling, nonlinear hydrodynamic processes, time–frequency distributions, vertical asymmetry, wave–wave interaction, wind–waves.**

I. Introduction

In a recent paper [1], we presented a new technique called the Stokes–Woodward technique that quantifies the amount of energy caused by nonlinear mode coupling. The starting point is a theorem stated by Woodward [2]–[4] that approximates the spectrum of frequency-modulated signals by the probability density function (pdf) of the instantaneous frequencies $P_{\dot{\phi}}$

$$S(f) \approx \frac{A^2}{2} P_{\dot{\phi}}(f - f_c) \tag{1}$$

with A being a constant amplitude and f_c a central or carrier frequency. A recent application of this theorem was successfully implemented in the study of delay and Doppler effects in bistatically reflected electromagnetic signals from the ocean surface [5]. We generalized this theorem in [1] to include joint amplitude (AM) and frequency (FM) modulations. A comparison was made of this new development with experimental data collected in a wind–wave tank. Two spectra were identified in addition to the traditional Fourier spectrum; the "bare" and the "dressed" spectra. The bare spectrum is obtained under no mode coupling conditions, or no wave–wave interactions in case of water waves. This can be understood as the occurrence of a family of random fundamental frequencies. The dressed spectrum, however, depicts the observable energy when mode coupling or wave–wave interactions are present and, therefore, can be interpreted as the energy augmentation due to a family of random harmonics. One other advantage of the Stokes–Woodward technique presented in [1] is to use only two-dimensional (2-D) joint pdfs (amplitude/frequency, asymmetry/ frequency) to quantify the amount of spectral energy due to nonlinearities as the use of a bispectral approach requires the evaluation of three-dimensional (3-D) histograms (asymetry/amplitude/frequency). At a minimum, this technique provides an explicit analytical expression of the spectrum that might be of use for further developments.

In the present letter, we show that the Stokes–Woodward technique can also be used to trace spectral energy in time. In this case, a new time–frequency technique is born, which detects wave occurrences in time. This transformation carries more information than a traditional wavelet transform. Three histograms estimated from the data enter into the study of local events. This time–frequency traceability is amenable to detect occurrences of mode coupling. Therefore, nonlinear packets can be detected in both frequency and time.

II. Summary of the Stokes–Woodward Technique

A. Statistical Modulation

As presented in [2], a first generalization of Woodward's theorem reduces to what we call the "bare" spectrum of a process jointly modulated in frequency and in amplitude under the condition of high modulation indexes

$$S_{\mathrm{bare}}(f) \approx \frac{1}{2} \int a^2 P_a(a, f)\, da \tag{2}$$

where $P_a(a, f)$ is the joint pdf of the instantaneous amplitude and frequency. This approximation is very practical and requires the estimation of a 2-D histogram from the available data. However, the assumption of high modulation index is equivalent to a slow modulation, this means that this practicality is gained at the expense of neglecting the temporal modulations of the amplitude and frequency at smaller time-scales than the dominant period. On the other hand, it is easily shown that the right-hand side of (2) is an exact formulation for the spectrum of a random process of this form

$$\eta(t) = a \cos(\omega t + \theta) \tag{3}$$

where a, ω, and θ are three time-independent random variables. The amplitude a and the pulsation $\omega = 2\pi f$ can be statistically dependent, while θ is a uniformly distributed phase and

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independent of these other variables. This requirement on the uniformity of the phase guarantees that the signal is statistically stationary and, therefore, that the autocorrelation and the spectral functions are univariate.

In other words, the high-index approximation that leads to (2) is equivalent to interpreting the temporal modulations as slow and random from one period to the next, while the instantaneous amplitude and frequency are assumed constant within a dominant time period. The "bare" subscript in (2) refers to the fact that the process in (3) has lost all nonlinearity or phase coupling of harmonics within one realization of the random variables.

B. Temporal and Statistical Modulation

The high-index limit in (2) is very illustrative. The time modulation present in (2) is now replaced by a random modulation as in (3) but expressed as follows:

$$\eta(t) = [a + \Delta a(t)] \cos [\omega t + \Delta\phi(t) + \theta] \tag{4}$$

where the time dependence is explicitly shown in addition to the implicit random dependence of all the parameters except the time variable. The key representation of our technique is in the time dependence of the modulation that can be expanded in Fourier series around the random frequency ω as

$$\Delta a(t) = \alpha_c \cos(\omega t + \theta) + \alpha_s \sin(\omega t + \theta) + \dots \tag{5a}$$

$$\Delta \phi(t) = \beta_c \cos(\omega t + \theta) + \beta_s \sin(\omega t + \theta) + \dots . \tag{5b}$$

The signal in (4) can be further expanded keeping only terms of linear order in the parameters α_c, α_s, β_c, and β_s to find a more poignant form given by (λ being a random variable)

$$\begin{aligned}\eta(t) = \lambda &+ a\cos[\omega t + \theta] \\ &+ \alpha \cos[2(\omega t + \theta)] + \beta \sin[2(\omega t + \theta)] + \dots\end{aligned} \tag{6}$$

where the (in time varying) coefficients α and β of the second harmonic terms in (6) are related to the coefficients in (5) (see [1]). We remind the reader that all parameters in (6) are random variables except the time variable. The random variables α and β explain the asymmetries of the waveform with respect to the horizontal and vertical axes, respectively, and hence nonlinearities. These asymmetries appear in a random manner on the scale of the period of the wave as depicted by the random amplitude a and random frequency ω.

As explained in [1], the total spectrum of the signal in (6) is then what we call the dressed spectrum and is given by

$$S_{\text{dressed}}(f) \approx S_{\text{bare}}(f) + S_{\text{NL}}(f) \tag{7}$$

which is the sum of the "bare" spectrum and a nonlinear contribution to the spectral energy [α and β as defined in (6)]

$$S_{\text{NL}}(f) = \frac{1}{2}\int \alpha^2 P_\alpha\left(\alpha, \frac{f}{2}\right) d\alpha + \frac{1}{2}\int \beta^2 P_\beta\left(\beta, \frac{f}{2}\right) d\beta. \tag{8}$$

The 2-D probability density distributions in (2) and in (8) are different and are to be estimated from the time series itself if they are not known *a priori*. The surface elevation time series provides, thanks to the zero-crossing technique, time series of α, β, and f the instantaneous frequency. From those time series, one can easily compute the number $N(\alpha_0, f_0)$ of data points corresponding to, $\alpha_0 < \alpha < \alpha_0 + \delta\alpha$ and $f_0 < f < f_0 + \delta f$. N being the number of points of the series in a and f we have

$$P_\alpha(\alpha_0, f_0) = \frac{N(\alpha_0, f_0)}{(N * \delta\alpha * \delta f)}. \tag{9}$$

Discretized pdfs are obtained this way. We associate this excess spectrum in (8) with terms added to the bare spectrum as contributing to the energy increase at higher frequencies due to the nonlinearities or mode coupling. This energy augmentation, therefore, provides the difference between the bare and the dressed spectra as originally introduced and discussed in [6]. The difference between bare and dressed spectra as given in (8) can also be assimilated with the bicoherence function of phase coupling as introduced by [7] and utilized by [8]. Indeed, energy of phase coupling to second order is another manifestation of the bispectrum defined as the Fourier transform of the skewness function using a third order cumulant. More details about the link between bispectral analysis and the Stokes–Woodward technique can be found in [1]. However, as mentioned in [1], the quantification of the bispectrum requires the estimation of two 3-D histograms whose evaluations are CPU intensive and can be very unstable.

V. CONCLUSION

An extension to our Stokes–Woodward technique is presented where the time dependence is added to the spectral formulation developed in [1]. Our original method consisted of generalizing Woodward's theorem by including random amplitude modulations in addition to frequency modulations. The original theorem stated that a good approximation of the energy spectrum of a frequency modulated signal is the probability density function of the instantaneous frequencies when the index of modulation is high. Our generalization simply starts by including the random amplitude modulation that yields a simple spectrum expressed as a single integral over the instantaneous amplitudes and the joint distribution of amplitude and frequency as shown in (2). It is noted that this spectrum is devoid of any nonlinearity or mode coupling because over the scale of a characteristic period, the wave is considered as simply harmonic (a sine wave). Asymmetries in the wave profile must be introduced in order to capture residual energy (8) not explained by the "bare" spectrum. To account for this residual energy, we have proposed a second generalization of Woodward's theorem that utilizes a Stokes-like waveform in which all the parameters are random except the time variable. The total spectrum is termed the "dressed" spectrum. Both spectra were compared to a standard Fourier spectrum in Fig. 1. Our current extension explores the time dependence of the spectral analysis. It is shown that the bare time–frequency distribution defined in (10) and illustrated in Fig. 2 is in good agreement with standard Wavelet analysis. The appearance of wave groups is detected. The novel dressed time–frequency distribution as defined in (12) provides a robust estimator of the importance of mode coupling and its time dependence. We believe that our Stokes–Woodward technique has broad relevance for understanding the physics of nonlinear processes even beyond the example of surface waves discussed in the present letter.